BIRZEIT UNIVERSITY

MATHEMATICS DEPARTMENT
MATH330, Second Hour Exam

- Name....Key.......... • Number.................. • Section..................

(Q1) [19 pts] Given the data

x	-4	-2	2	4
y	0	1	1	0

1. Use all the nodes to find Lagrange's polynomial $p_n(x)$.
2. Use all the nodes to find Newton's polynomial $p_n(x)$.
3. Suppose that $f^{(4)}(x) \in [-5, 2]$, $\forall x \in [-4, 4]$. Use the nodes in the table to find the upper bound of the error $|f(0) - p_n(0)|$.
4. Find $f[-2, 2, 4]$.

1) $P_3(x) = 0 + \dfrac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$

$+ \dfrac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + 0$

4

$P_3(x) = \dfrac{(x+4)(x-2)(x-4)}{(2)(-4)(-6)} + \dfrac{(x+4)(x+2)(x-4)}{(6)(4)(-2)}$

$= \dfrac{(x-2)(x^2-16)}{48} - \dfrac{(x+2)(x^2-16)}{48} = -\dfrac{(x^2-16)}{12}$

2

2) Newton poly. = Lagrange poly.

$\Rightarrow P_3(x) = \dfrac{16-x^2}{12}$

6

OR: OR: $P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$
$+ a_3(x-x_0)(x-x_1)(x-x_2)$

x_k	y_k	1st D.D	2nd D.D	3rd D.D
-4	0	—	—	—
-2	1	$\frac{1}{2}$	—	—
2	1	0	$-\frac{1}{12}$	—
4	0	$-\frac{1}{2}$	$-\frac{1}{12}$	0

each a_i 1 point

$$\Rightarrow P_3(x) = 0 + \tfrac{1}{2}(x+4) + -\tfrac{1}{12}(x+4)(x+2) + 0$$
$$= \frac{16-x^2}{12}$$ 2

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3) $$E_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(4)}(c)$$ 1

$$E_3(0) = \frac{(4)(2)(-2)(-4)}{24} f^{(4)}(c) = \frac{64}{24} f^{(4)}(c)$$ 1

but $|f^{(4)}(x)| \leq 5 \quad , \forall x \in [-4,4]$ 1

$$\Rightarrow |E_3(0)| \leq \frac{64}{24}(5) = \frac{40}{3}$$ 1

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4) $$f[-2,2,4] = \frac{\frac{f(4)-f(2)}{4-2} - \frac{f(2)-f(-2)}{2-(-2)}}{4+2}$$ 3

$$= \boxed{\frac{-1}{12}}$$

(Q2) [8 pts] Given $f(x) = \ln(x+1)$, $x \in [3.2, 3.8]$. Use equally space nodes to find the upper bounds for the interpolation errors $E_1(x), E_2(x), E_3(x)$.

$$f'(x) = \frac{1}{x+1}, \quad f''(x) = \frac{-1}{(x+1)^2}, \quad f'''(x) = \frac{2}{(x+1)^3}, \quad f(x) = \frac{-6}{(x+1)^4}$$

1 1 1 1

$$|E_1(x)| \leq \frac{h^2 M_2}{8} = \frac{(0.6)^2 \, Max\,|f''(x)|}{8} = \frac{(0.6)^2(0.05669)}{8} = 0.00255$$

1 1 1

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} = \frac{(0.3)^3 \, Max\,|f^{(3)}(x)|}{9\sqrt{3}} = \frac{(0.3)^3(0.02699)}{9\sqrt{3}} = 4.67\times10^{-5}$$

1 1 1

$$|E_3(x)| \leq \frac{h^4 M_4}{24} = \frac{(0.2)^4 \, Max\,|f^{(4)}(x)|}{24} = \frac{(0.2)^4(0.01928)}{24} = 1.28\times10^{-1}$$

(Q3) [8 pts] If the function below is the clamped cubic spline for $f(x)$ in $[0, 2]$.

$$S(x) = \begin{cases} S_0(x) = 10 - 10x + Ax^2 + 2x^3 & ; \ 0 \leq x \leq 1 \\ S_1(x) = B + 8(x-1)^2 + (x-1)^3 & ; \ 1 \leq x \leq 2 \end{cases}$$

Find the constants A, B. Then find $f'(0)$ and $f'(2)$.

$$S'(x) = \begin{cases} -10 + 2Ax + 6x^2, & 0 \leq x \leq 1 \\ 16(x-1) + 3(x-1)^2, & 1 \leq x \leq 2 \end{cases}$$

$$S_0'(1) = S_1'(1) \iff -4 + 2A = 0 \Rightarrow \boxed{A = 2}$$ 2

$$S_0(1) = S_1(1) \iff A + 2 = B$$

$$\boxed{4 = B}$$ 2

$$f'(0) = S_0'(0) = \boxed{-10}$$ 2

$$f'(2) = S_1'(2) = 16 + 3 = \boxed{19}$$ 2

(Q4) [8 pts] Show that the natural cubic spline $g(x)$ that interpolates the function $f(x)$ at the points $(-2,3),(0,1),(2,7)$ is given by

$$g(x)=\begin{cases}1+x+x^2+\frac{1}{4}x^2(2+x), & x\in[-2,0]\\ 1+x+x^2+\frac{1}{4}x^2(2-x), & x\in[0,2].\end{cases}$$

$$g(x)=\begin{cases}1+x+x^2+\frac{1}{2}x^2+\frac{1}{4}x^3, & -2\le x\le 0\\ 1+x+x^2+\frac{1}{2}x^2-\frac{1}{4}x^3, & 0\le x\le 2\end{cases}$$

$$g(x)=\begin{cases}1+x+\frac{3}{2}x^2+\frac{x^3}{4}, & -2\le x\le 0\\ 1+x+\frac{3}{2}x^2-\frac{x^3}{4}, & 0\le x\le 2\end{cases}$$

$$g'(x)=\begin{cases}1+3x+\frac{3x^2}{4}, & -2\le x\le 0\\ 1+3x-\frac{3x^2}{4}, & 0\le x\le 2\end{cases}$$

$\boxed{1}$ $g_0(x_0)=y_0 \Rightarrow g_0(-2)=1-2+6-2=3=f(-2)$

$\boxed{2}$ $g_1(x_1)=y_1 \Rightarrow g_1(0)=1=f(0)$

$\boxed{3}$ $g_1(x_2)=y_2 \Rightarrow g_1(2)=1+2+6-2=7=f(2)$

$\boxed{4}$ $g_0(0)=1=g_1(0)$

$\boxed{5}$ $g_0'(0)=1=g_1'(0)$

$\boxed{6}$ $g_0''(0)=3=g_1''(0)$

$\boxed{7}$ $g_0''(-2)=0=3-3$

$\boxed{8}$ $g_1''(2)=3-3=0$

$$g''(x)=\begin{cases}3+\frac{3}{2}x, & [-2,0]\\ 3-\frac{3}{2}x, & [0,2]\end{cases}$$

(1 point each)

OR:

(Q4) [8 pts] Show that the natural cubic spline $g(x)$ that interpolates the function $f(x)$ at the points $(-2,3),(0,1),(2,7)$ is given by

$$g(x)=\begin{cases} 1+x+x^2+\frac{1}{4}x^2(2+x), & x\in[-2,0] \\ 1+x+x^2+\frac{1}{4}x^2(2-x), & x\in[0,2]. \end{cases}$$

$$g(x)=\begin{cases} a_0(x+2)^3+b_0(x+2)^2+c_0(x+2)+d_0, & -2\le x\le 0 \\ a_1(x)^3+b_1x^2+c_1x+d_1, & 0\le x\le 2 \end{cases}$$

1) $g_0(x_0)=y_0 \Rightarrow \boxed{d_0=3}$

1 point each constant

$g_1(x_1)=y_1 \Rightarrow \boxed{d_1=1}$

$g_1(x_2)=y_2 \Rightarrow 8a_1+4b_1+2c_1+d_1=7$

$g_0(x_1)=g_1(x_1) \Rightarrow 8a_0+4b_0+2c_0+d_0=d_1=1$

$$g'(x)=\begin{cases} 3a_0(x+2)^2+2b_0(x+2)+c_0, & -2\le x\le 0 \\ 3a_1x^2+2b_1x+c_1, & 0\le x\le 2 \end{cases}$$

$$g''(x)=\begin{cases} 6a_0(x+2)+2b_0, & -2\le x\le 0 \\ 6a_1x+2b_1, & 0\le x\le 2 \end{cases}$$

$g_0'(x_1)=g_1'(x_1) \Leftrightarrow 12a_0+4b_0=c_1$

$g_0''(x_1)=g_1''(x_1) \Leftrightarrow 12a_0+2b_0=2b_1$

inally, $g_0''(x_0)=0=2b_0=0$ & $g_1''(x_2)=0=12a_1+2b_1=0$

$\Rightarrow$ $a_0=\frac{1}{4}$, $b_0=0$, $c_0=-2$, $d_0=3$

$a_1=-\frac{1}{4}$, $b_1=\frac{3}{2}$, $c_1=1$, $d_1=1$

$$g(x) = \begin{cases} \frac{1}{4}(x+2)^3 + -2(x+2) + 3 \\ -\frac{1}{4}(x)^3 + \frac{3}{2}x^2 + x + 1 \end{cases}$$

$$= \begin{cases} 2 + 3x + \frac{3x^2}{2} + \frac{x^3}{4} - 2x - 4 + 3 \\ -\frac{1}{4}x^3 + \frac{3}{2}x^2 + x + 1 \end{cases}$$

$$= \begin{cases} 1 + x + \frac{3}{2}x^2 + \frac{x^3}{4}, & -2 \leq x \leq 0 \\ 1 + x + \frac{3}{2}x^2 - \frac{1}{4}x^3, & 0 \leq x \leq 2. \end{cases}$$

$$= \begin{cases} 1 + x + \frac{6}{4}x^2 + \frac{x^3}{4}, & -2 \leq x \leq 0 \\ 1 + x + \frac{\cancel{6}}{4}x^2 - \frac{1}{4}x, & 0 \leq x \leq 2. \end{cases}$$

(Q5) [11 pts] Find the normal equations of the least-square curve of the form $f(x) = A + e^{Bx} + C\sin x$.

$$E(A,B,C) = \sum \left(A + e^{Bx_k} + C\sin x_k - y_k\right) \quad 2$$

$$\frac{\partial E}{\partial A} = \sum 2\left(A + e^{Bx_k} + C\sin x_k - y_k\right)\cdot 1 = 0 \quad 3$$

$$\frac{\partial E}{\partial B} = \sum 2\left(A + e^{Bx_k} + C\sin x_k - y_k\right)\cdot e^{Bx_k}\cdot x_k = 0 \quad 3$$

$$\frac{\partial E}{\partial C} = \sum 2\left(A + e^{Bx_k} + C\sin x_k - y_k\right)\sin x_k = 0. \quad 3$$

(Q6) [6 pts] Find the truncation error when using the difference formula $f'(x_0) \approx \dfrac{f(x_0+h) - f(x_0)}{h}$.

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2 f''(c)}{2!} \quad 3$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h^2 f''(c)}{h\, 2!} \quad 3$$

$$\Rightarrow \quad E_{trun}(f) = \frac{-h f''(c)}{2!}$$

OR:

$$E(f) = \frac{(x-x_0)(x-x_1) f''(c)}{2!} \quad 2$$

$$E'(f) = \left[(x-x_0)+(x-x_1)\right] \frac{f''(c)}{2!} \quad 2$$

$$E'(x_0) = \left[(x_0 - x_1)\right] \frac{f''(c)}{2!} = \frac{-h f''(c)}{2!}. \quad 2$$

(Q7) [10 pts] Given the points $(2, 2.5), (3, 5), (4, 7)$. Use linearization to find the best fitting curve of the form $y = A\cos(x) + B\ ln(x)$.

$$y = A\cos(x) + B\ln(x)$$

$$\frac{y}{\ln(x)} = A\frac{\cos(x)}{\ln(x)} + B \qquad ②$$

$$\bar{Y} = A\bar{X} + B$$

Normal equations for line : $\sum A X_k^2 + \sum B X_k = \sum X_k Y_k$

$\sum A X_k + \sum B = \sum Y_k.$

x	y	$\bar{X}$	$\bar{Y}$	$\bar{X}^2$	$\bar{X}\bar{Y}$
2	2.5	-0.6	3.61	0.36	-2.166
3	5	-0.9	4.55	0.81	-4.095
4	7	-0.47	5.05	0.2209	-2.3735

④

$\Rightarrow$ $1.3909A + -1.97B = -8.6345$ ②

$-1.97A + 3B = 13.21$

Solve for A & B $\Rightarrow$ $A = 0.411926$ ②

$B = 4.67383$

OR:

$$y = A\cos(x) + B\ln(x)$$

$$\frac{y}{\cos(x)} = A + B\,\frac{\ln(x)}{\cos(x)}.$$

$$B\sum X_k^2 + A\sum X_k = \sum X_k Y_k$$

$$B\sum X_k + 3A = \sum Y_k.$$

x_k	y_k	$\bar{X}_k$	$\bar{Y}_k$	$\bar{X}_k^2$	$\bar{X}_k\bar{Y}_k$
2	2.5	-1.67	-6.61	2.79	10.04
3	5	-1.11	-5.05	1.23	5.61
4	7	-2.12	-10.71	4.49	22.71

$$8.51B - 4.9A = 38.36$$

$$-4.9B + 3A = -21.77$$

$$\Rightarrow \quad A \approx 1.78$$

$$B = 5.53$$

(Q8) [10 pts] Given the difference formula below.

$$f'(x_0) = \frac{-8f_0 + 9f_1 - f_3}{6h} + \frac{h^2 f'''(c)}{2}, \text{ where } f_0 = f(x_0), f_1 = f(x_0 + h), f_3 = f(x_0 + 3h)$$

(a) Derive this formula using Taylor's expansion.

(b) Use this formula with the points $(2, -1), (2.5, 4), (3, 2), (3.5, 1), (4, 5)$ to estimate $f'(2)$ and $f'(2.5)$.

$$f_1 = f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2 f''(x_0)}{2!} + \frac{h^3}{3!} f^{(3)}(c) \quad 2$$

$$f_3 = f(x_0+3h) = f(x_0) + 3h f'(x_0) + \frac{9h^2}{2!} f''(x_0) + \frac{27h^3}{3!} f^{(3)}(c) \quad 2$$

$$9f_1 - f_3 = 8f_0 + 6h f'(x_0) - \frac{18h^3}{3!} f^{(3)}(c) \quad 1$$

$$\Rightarrow f'(x_0) = \frac{-8f_0 + 9f_1 - f_3}{6h} + \frac{h^2}{2} f^{(3)}(c) \quad 1$$

b) $$f'(2) = \frac{-8f(2) + 9f(2.5) - f(3.5)}{6(0.5)} \quad 1$$

$$= \frac{8 + 36 - 1}{3} = 14.333 \quad 1$$

$$f'(2.5) = \frac{-8f(2.5) + 9f(3) - f(4)}{6(0.5)} \quad 1$$

$$= \frac{-32 + 18 - 5}{3} = -6.3333 \quad 1$$